A note on the anisotropy and fabric of highly porous materials

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The dependence of the orthotropic elastic constants of a highly porous material upon the stereological parameters characterizing the anisotropy of the porous microstructure has been considered in two recent papers in this journal. In the first paper [1] dimensional arguments were employed to develop to a relationship between ratios of the orthotropic elastic constants and ratios of the mean intercept lengths for a class of cell wall bending models of highly porous materials. In the second paper [2] the general tensorial form of the relationship between the orthotropic elastic constants and the mean intercept length was described without reference to a specific form or type of porous microstructure. The purpose of this note is to observe that the particular relationships obtained from the class of cell wall bending models used in the first paper are proper special cases of the general relationships given in the second paper.

1. Introduction

A relationship between the fourth-rank tensor of elastic constants of a porous, anisotropic, linear elastic material and stereological parameters characterizing the anisotropy of the microstructure of the material was presented by Cowin [3]. In addition to the porosity, the stereological parameter characterizing the anisotropy of the microstructure was called the fabric ellipsoid or fabric tensor of the material. In the present work the fabric ellipsoid, represented by a symmetric second-rank tensor, characterizes the geometric arrangement of the porous material microstructure by a directed mean intercept length measure. In developing this relationship between elastic constants and fabric [3], it was assumed that the matrix material of the porous elastic solid was isotropic and that the anisotropy of the porous elastic solid was completely determined by the fabric ellipsoid or tensor. It was then shown that the material symmetries of orthotropy, transverse isotropy and isotropy correspond to the cases of three, two and one distinct eigenvalues of the fabric tensor, respectively. In terms of the fabric ellipsoid, these cases correspond to cases of an ellipsoid with three, two and no unequal axes. The characteristics of the fabric measure are described in the next section, and the formulae for the dependence of the orthotropic elastic constants upon fabric are given in Section 3. The formulae for the dependence of the orthotropic elastic constants upon fabric for the class of open cell foam models considered by Huber and Gibson [1] are summarized and converted into the notation of this paper in Section 4. It is then shown that the formulae presented by Huber and Gibson, and obtained from a specific class of structural models of an open celled foam, are special cases of the formulae given by Cowin [3] and Turner and Cowin [2] and based on a general, not cell wall bending specific, model of an anisotropic porous material. This coincidence reinforces both the general approach of Cowin [3] and the cell wall bending model approach of Huber and Gibson [1]. This matter is considered further in the final discussion, Section 5.

2. Fabric

The representation of the degree of symmetry and the type of symmetry for orthotropic elastic symmetries and higher elastic symmetries by a single symmetric second-rank tensor representing the material microstructure have previously been explored [2–4]. The generic name used by many for this tensor representing the material microstructure is the fabric tensor. The exact definition of the fabric tensor varies with the material being considered and with the investigator. Some of the definitions of the fabric tensors for various porous geological and biological materials are described by Turner and Cowin [2]. In this paper the definition of fabric as a directed mean intercept length measure is employed. This definition of fabric is described by Cowin [5, 6] and Turner and Cowin [2]; it is based on work of Harrigan and Mann [7].

Let H_i, $i = 1, 2, 3$, denote the lengths of the three axes of the fabric ellipsoid or, equivalently, the eigenvalues of the fabric tensor $\boldsymbol{H}$. In the principal coordinate system of $\boldsymbol{H}$, the matrix of tensor components has the form

$$\boldsymbol{H} = \begin{bmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{bmatrix} \tag{1}$$

The parameter H_i represents the mean intercept length in the coordinate direction x_i, where x_i is

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a principal direction of $\boldsymbol{H}$. In this work only the diagonal form of $\boldsymbol{H}$ will be considered. Each of the H_i satisfies the cubic equation

$$H_i^3 - \mathrm{I}H_i^2 + \mathrm{II}H_i - \mathrm{III} = 0, \quad i = 1, 2, 3 \tag{2}$$

where I, II, and III represent the invariants of the fabric tensor $\boldsymbol{H}$. The invariants are related to the principal components H_i, by the formulae

$$\begin{aligned} \mathrm{I} &= H_1 + H_2 + H_3 \\ \mathrm{II} &= H_1H_2 + H_1H_3 + H_3H_2 \\ \mathrm{III} &= H_1H_2H_3 \end{aligned} \tag{3}$$

These invariants are related to the traces of $\boldsymbol{H}$, $\boldsymbol{H}^2$ and $\boldsymbol{H}^3$ by the formulae obtained, for example, by Ericksen [8]

$$\begin{aligned} \mathrm{tr}\boldsymbol{H} &= H_1 + H_2 + H_3 = \mathrm{I} \\ \mathrm{tr}\boldsymbol{H}^2 &= H_1^2 + H_2^2 + H_3^2 = \mathrm{I}^2 - 2\mathrm{II} \\ \mathrm{tr}\boldsymbol{H}^3 &= H_1^3 + H_2^3 + H_3^3 = \mathrm{I}^3 - 3\mathrm{I}(\mathrm{II}) + 3\mathrm{III} \end{aligned} \tag{4}$$

3. The dependence of the orthotropic elastic constants upon fabric

There are nine distinct orthotropic elastic constants in a symmetrical coordinate system. The particular nine elastic constants considered here are called the technical elastic constants and consists of three Young's moduli, three shear moduli, and six Poisson's ratios (only three of which are independent). The Young's moduli are denoted by E_i, $i = 1, 2, 3$; the shear moduli by G_{ij} and the Poisson's ratios by ν_{ij} where $i, j = 1, 2, 3$ and $i \neq j$. The interrelationships between the Young's moduli and Poisson's ratios are given by

$$E_j\nu_{ij} = E_i\nu_{ji}, \quad i \neq j, \text{ no sum on } i, j. \tag{5}$$

It was shown by Cowin [3] (see also [2]) that these elastic constants are related to the fabric components by

$$\frac{1}{G_{ij}} = c_1 + c_2(H_i + H_j) + c_3(H_i^2 + H_j^2), \quad i \neq j \tag{6}$$

$$\frac{1}{E_i} = a_1 + 2c_1 + d_1 + 2(a_2 + 2c_2 + d_2)H_i + (2a_3 + b_1 + 4c_3 + d_3)H_i^2 \tag{7}$$

$$-\frac{\nu_{ij}}{E_i} = -\frac{\nu_{ji}}{E_j} = a_1 + a_2(H_i + H_j) + a_3(H_i^2 + H_j^2) + b_1H_iH_j + b_2H_iH_j(H_i + H_j) + b_3H_i^2H_j^2 \quad i \neq j, \; i, j = 1, 2, 3 \tag{8}$$

where the a_i, b_i, c_i and d_i, $i = 1, 2, 3$, are functions of I, II and III and some measure of the porosity of the material. The coefficients d_i, $i = 1, 2, 3$, are functions of the coefficients b_i and the invariants I, II, and III,

$$d_1 = (2b_2 + b_3\mathrm{I})\mathrm{III}, \quad 2d_2 = b_3(\mathrm{III} - \mathrm{I}(\mathrm{II})) - 2b_2\mathrm{II}, \quad d_3 = 2b_2\mathrm{I} + b_3(\mathrm{I}^2 - \mathrm{II}) \tag{9}$$

In the representation for E_i the characteristic Equation 2 has been used to express the third- and fourth-order terms in H_i given in the formulae in Cowin [3] and Turner and Cowin [2] in terms of the second- and first-order terms; the coefficients d_i arise from this algebraic manipulation. The points considered in this note do not concern the porosity dependence of the coefficients a_i, b_i, c_i and d_i, $i = 1, 2, 3$ directly; this dependence is important and always permitted, however.

4. The anisotropy of foams

In [1] the elements of mechanics of materials were applied to a class of cell wall bending models of an elastically orthotropic foam to construct the algebraic relationship between ratios of the three Young's moduli, the ratios of the three shear moduli, and the principal mean intercept lengths. The unit cells in the structural model were connected to adjacent cells at the cell midpoint to develop a bending response of the structure. Examples of these results are the ratios G_{12}/G_{23} and E_3/E_1 as functions of the mean intercept lengths H_i, $i = 1, 2, 3$

$$\frac{G_{12}}{G_{23}} = \frac{(H_2 + H_3)}{(H_2 + H_1)} \tag{10}$$

and

$$\frac{E_3}{E_1} = \frac{H_3^2(1 + H_2^3/H_1^3)}{H_1^2(1 + H_2^3/H_3^3)} \tag{11}$$

respectively. The notation employed by Huber and Gibson [1] for these results is slightly different: the ratio R_{ij}, where $i, j = 1, 2, 3$ and $i \neq j$, is used in place of H_i/H_j. Straightforward algebraic manipulations, combined with the use of the notation introduced in the third equation of (4), namely $\mathrm{tr}\boldsymbol{H}^3$ for $(H_1)^3 + (H_2)^3 + (H_3)^3$, permits (11) to be rewritten in the form

$$\frac{\dfrac{1}{E_1}}{\dfrac{1}{E_3}} = \frac{\dfrac{1}{H_1^5(\mathrm{tr}\boldsymbol{H}^3 - H_1^3)}}{\dfrac{1}{H_3^5(\mathrm{tr}\boldsymbol{H}^3 - H_3^3)}}. \tag{12}$$

These results suggest that the reciprocals of G_{ij} and E_i may be represented by

$$\frac{1}{G_{ij}} = G(H_i + H_j), \quad i \neq j \tag{13}$$

and

$$\frac{1}{E_i} = \frac{E}{H_i^5(\mathrm{tr}\boldsymbol{H}^3 - H_i^3)} \tag{14}$$

where G and E represent arbitrary functions of I, II, III and the porosity.

The representations 13 and 14 for G_{ij} and E_i are special cases of the representations given above by Equations 6 and 7, respectively. It is easy to see that Equation 13 is a special case of Equation 6 because the two formulae coincide if one sets c_1 and c_3 equal to zero and c_2 equal to G in the formula 6. A casual inspection suggests that it might not be possible to obtain coincidence between Equations 14 and 7, but such is not the case. The coincidence can be seen in the following way: formally divide the denominator into the numerator of Equation 14. The result is an infinite series in H_i, the nth term being of the form $(H_i)^n$ times a scalar valued function of I, II and III. (Recall from Equation 4 that tr$\boldsymbol{H}^3$ can be expressed in terms of I, II and III.) All terms of order higher than $(H_i)^2$ can then be eliminated from the infinite series by repeated use of Equation 2. The result is then a specific form of Equation 7, and it has been established that the results, Equations 13 and 14, are special cases of Equations 6 and 7, respectively.

5. Discussion

There are several interesting points concerning the simple algebraic observations described above. The observations represent positive and reinforcing interaction between the development of a general theory and the development of specific structural models for the anisotropy of highly porous materials. The results that were common to both the general model and the cell wall bending model are the following.

(a) The orthotropic elastic Young's moduli E_i depend directly upon H_i and the elementary symmetric functions I, II and III of H_1, H_2, H_3. This means that the influence of the components of $\boldsymbol{H}$ other than H_i only occurs through the elementary symmetric functions I, II and III.

(b) The orthotropic shear moduli G_{ij} depend directly upon H_i, H_j, I, II and III. This means that the influence of the component of $\boldsymbol{H}$ other than H_i and H_j occurs only through I, II and III. Both approaches predict a dependence of the inverse of G_{ij} upon the sum $H_i + H_j$.

The notation employed in Equations 6, 7 and 8 for the relationships between the orthotropic elastic constants and the mean intercept length measures presents these results in a relatively concise form, emphasizing the appropriate symmetries in the functional dependencies. In order to obtain the corresponding results for transversely isotropic symmetry one has only to set two of the H_i equal. The number of elastic coefficients then reduces from nine to five. Note that, if one takes H_1 and H_2 to be equal, then E_1 and E_2, G_{13} and G_{23}, ν_{12} and ν_{21}, ν_{13} and ν_{23}, and ν_{31} and ν_{32} are all equal, G_{12} equals E_1 divided by $2(1 + \nu_{12})$, and $E_3\nu_{13}$ equals $E_1\nu_{31}$.

In the development above of the correspondence between the general approach of Cowin [3] and the cell wall bending model approach of Huber and Gibson [1], it is assumed that the spatial dsitribution of the directed mean intercept lengths of the cell wall bending model approach will form a second-rank tensor or, equivalently, an ellipsoid. The arguments of Huber and Gibson [1] apply to a class of models, a class that includes geometrically regular rectangular structural cell models of the type illustrated therein, as well as less regular models. The consideration of the mean intercept length as a function of spatial direction for idealized linear plane geometric structures reported by Tozeren and Skalak [9] shows that the directed mean intercept lengths of the geometrically very regular model types considered by Huber and Gibson [1], for example, the rectangular structural cell model type, will not form an ellipsoid, but rather a polyhedron with three orthogonal axes of symmetry. Along each orthogonal axis of symmetry the half-dimension of the polyhedron will be equal to one of the mean intercept lengths and the eight octants of the polyhedron will be identical, The exterior boundary of each octant will be a mosaic of polygons. It is also suggested by the results of Tozeren and Skalak [9] that, as the precise regularity of the cell model tends to the irregularities of natural porous materials, the polyhedron representing the spatial distribution of the directed mean intercept lengths will approach more closely an ellipsoid. In the analysis presented above it is assumed that even the polyhedron representing the spatial distribution of the directed mean intercept lengths of the geometrically very regular model types considered by Huber and Gibson [1], for example, the rectangular structural cell model type, can be approximated by an ellipsoid. This approximation may be accomplished, for example, by a least squares fit of an ellipsoid to the actual polyhedron representing the spatial distribution of the directed mean intercept lengths.

Finally, it is noted that formulae are given by Huber and Gibson [1] for the dependence upon the ratios of the mean intercept lengths in different directions of both the ratios of elastic collapse stresses in different directions and the plastic collapse stresses in different directions. The methods described above can be employed to show that these results are consistent with the theory of anisotropic strength dependence upon fabric presented by Cowin [4]. Both specific results are consistent with the same general theory; the general theory is like a "theory of strength" and does not distinguish between yield and ultimate stress nor between elastic and plastic collapse stresses.

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